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Multidimensional pattern formation has an infinite number of constants of motion

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Extending our previous work on two-dimensional growth for the Laplace equation [M. B. Mineev, Physica D **43**, 288 (1990)] we study here *multidimensional* growth for *arbitrary elliptic* equations, describing inhomogeneous and anisotropic pattern-formation processes. We find that these nonlinear processes are governed by an infinite number of conservation laws. Moreover, in many cases all *dynamics of the interface can be reduced to the linear time dependence of only one "moment"* M_0, which corresponds to the changing volume, *while all higher moments* M_l *are constant in time.* These moments have a purely geometrical nature, and thus carry information about the moving shape. These conserved quantities [Eqs. (7) and (8) of this article] are interpreted as coefficients of the multipole expansion of the Newtonian potential created by the mass uniformly occupying the domain enclosing the moving interface. Thus the question of how to recover the moving shape using these conserved quantities is reduced to the classical inverse potential problem of reconstructing the shape of a body from its exterior gravitational potential. Our results also suggest the possibility of controlling a moving interface by appropriately varying the location and strength of sources and sinks.

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Many seemingly different pattern formation processes have much in common, both in their mathematical description and in their physical behavior. Among them are the famous Stéfan problem (freezing of liquid), flow through porous media, the Rayleigh-Taylor instability, electrodeposition of metals, corrosion, combustion, growth of bacterial colonies, dynamics of earth cracks, diffusion-limited aggregation (DLA), etc. The common feature shared by these processes is the existence of an evolving interface. The problem of the evolution of the interface in these processes attracts considerable attention (see, for example, Ref. [1]) both because of its great practical importance and because of its connections with such fields as dynamical chaos, nonequilibrium physics, and fractal growth (see DLA [2]).

A general scheme for these processes is as follows. There is a linear partial differential equation (PDE) (frequently of the second order) for the scalar field determining the process. For example, this is the diffusion equation for the Stéfan problem and the Laplace or Helmholtz equations for electrodeposition. This scalar field is temperature in the Stéfan problem, pressure in flows through porous media, concentration of the nutrient for bacterial growth, electrostatic potential in electrodeposition, probability of the next jump in DLA, etc. Appropriate boundary conditions are imposed both on the moving part of the boundary (the interface) and the nonmoving part of the boundary (the outer walls). In addition, a law of interface motion is given in terms of the local behavior of the main scalar field. (Typically the local velocity of the interface is proportional to the gradient of the scalar field near the interface.) The main question is "what is the evolution of the interface?"

It is remarkable that some of the problems mentioned above are exactly solvable in two dimensions [3–5], despite the nonlinearity of these processes. These problems were solved with the help of time-dependent conformal mapping, which cannot be extended to three dimensions (3D), except for a few trivial cases. In 2D it was found that such processes as two-phase flows in porous media, electrodeposition, and slow solidification in a supercooled liquid or from a supersaturated binary solution are governed by an infinite number of constants of motion, which were obtained explicitly in several special cases [3–5]. These constants of motion are related to the conserved moments proposed by Richardson [6]. This invariance is quite subtle and disappears when realistic physical perturbations such as surface tension or random noise are added.

In the 3D case very few exact analytical results are known: a constant-velocity paraboloid [7] and a self-similarly growing ellipsoid [8]. The only known way to obtain these solutions is by using one of the 11 coordinate systems in which the 3D Laplace equation is separable [10]. One then considers level surfaces as moving interfaces. It is clear that this method does not work when the shapes are time dependent. The traditional attitude is that the main obstacle in obtaining nonperturbative exact results in 3D is the lack of (nontrivial) conformal mappings unlike in the 2D case. But is not this statement too strong?

This article is a natural extension of previous work [3] to more general and realistic *multidimensional growth* problems; and not only for the Laplace equation as was done in [3–5], but also for *arbitrary elliptic equations* describing, for example, inhomogeneous and anisotropic diffusion in solidification, inhomogeneous dielectric functions and screening in electrodeposition, and inhomogeneous viscosity for flows through porous media. It turns out that these nonlinear processes also possess remarkable properties (an infinite number of conservation laws) similar to the ones mentioned in [3,6] for the 2D Laplacian case, and these properties do not depend on the dimension of the process considered. Thus we show that, contrary to the traditional attitude, *we do not need a conformal mapping for this invariance.* Rather this invariance originates from the more general property: *the elliptic nature of the equation for the scalar field.*

Let us state now the following D-dimensional problem:

$$L(u)=\text{div}[p(\mathbf{r})\text{grad}u(\mathbf{r})]+q(\mathbf{r})u(\mathbf{r})=0 \tag{1}$$

for $\mathbf{r}\in B\subset\mathbb{R}^D$, where the domain B is bounded by the nonmoving exterior boundary Σ, and by the moving interior boundary $\Gamma(t)$ (t is time), which is the interface separating the domains B and A. An interior domain A contains the origin and is surrounded by the moving interface $\Gamma(t)$, as shown in Fig. 1. Here p and q are given functions of $\mathbf{r}=(x_1,x_2,\ldots,x_D)$. The boundary conditions imposed on u are

$$\partial_n u|_\Sigma=G(\Sigma)\ , \tag{2}$$

$$u(\Gamma(t))=0\ . \tag{3}$$

The left-hand side (lhs) of (2) means the normal component of grad(u) evaluated at Σ.

There can also exist pointlike sources and sinks in the domain B located at $\mathbf{r}_k$ and having strengths s_k ($k=1,2,\ldots,N$), so that near $\mathbf{r}_k$, $u(\mathbf{r})$ diverges and satisfies

$$u=s_k/|\mathbf{r}-\mathbf{r}_k|^{D-2}+\text{smooth function} \tag{4}$$

if $D>2$, or

$$u=s_k\ln|\mathbf{r}-\mathbf{r}_k|+\text{smooth function}$$

if $D=2$. The law of motion of $\Gamma(t)$ is

$$v_n=-p(\mathbf{r})\partial_n u|_{\Gamma(t)}\ , \tag{5}$$

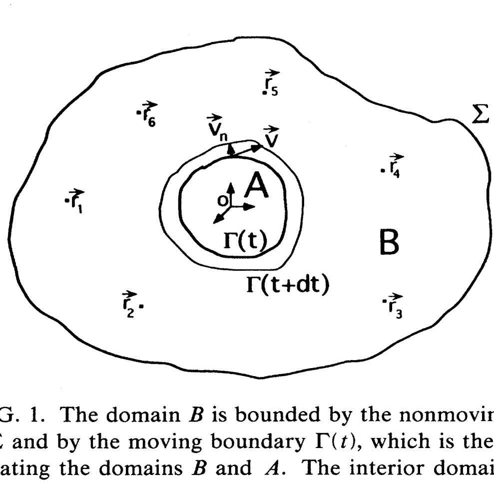

FIG. 1. The domain B is bounded by the nonmoving boundary Σ and by the moving boundary $\Gamma(t)$, which is the interface separating the domains B and A. The interior domain A contains the origin. The interface $\Gamma(t)$, depicted at times t and $t+dt$, moves with velocity $\mathbf{v}$ (and normal velocity $\mathbf{v}_n$). Also shown are pointlike sources or sinks, which may exist in the domain located at $\mathbf{r}_k$ and having strengths s_k ($k=1,2,\ldots,N$). In this figure, $N=6$.

where the lhs is the normal component of the velocity of $\Gamma(t)$.

Equations (1) and (5) together with the boundary conditions (2)–(4) complete the mathematical description of the motion of $\Gamma(t)$. If, for example, $D=3$, $p(\mathbf{r})=\text{const}$, and $q(\mathbf{r})=0$, this describes slow solidification or two-phase flow in homogeneous porous media.

In this paper, we show that, in spite of the complexity and nonlinearity of the processes described by Eqs. (1)–(5), these processes are governed by an infinite number of conservation laws. Namely, if the outer boundary Σ is very far from the origin there is an infinite set of numbers C_l ($l=D-1,D,D+1,\ldots,\infty$) defined as

$$C_l\equiv\frac{dM_l}{dt}\equiv\frac{d}{dt}\left[\int_B \psi_l d^D\mathbf{r}\right]\ , \tag{6}$$

which are conserved during the evolution of the hypersurface $\Gamma(t)$, and equal

$$C_l=2\frac{\pi^{D/2}}{\Gamma(D/2)}\sum_{k=1}^{N} s_k p(\mathbf{r}_k)\psi_l(\mathbf{r}_k)\ . \tag{7}$$

(We do not consider here the passing of the interface through the singularities.)

Here, $\Gamma(n)$ is the gamma function [*not* the interface $\Gamma(t)$] and ψ_l is the arbitrary solution of $L(u)=0$, which decays at infinity no slower than r^{-l} and has singularity only at the origin. The functions ψ_l as well as quantities C_l and M_l are labeled in general by more than one number (which is l here). See, for example, Eq. (11) below. But for simplicity and without the loss of generality we drop all labels except the l almost everywhere. If $q(\mathbf{r})=0$ we have one more conserved quantity: C_0, which is the rate of the change of the volume of B when L is Laplacian, and which satisfies

$$C_0=\frac{dM_0}{dt}=\frac{d}{dt}\left[\int_B d^D\mathbf{r}\right]$$

$$=2\frac{\pi^{D/2}}{\Gamma(D/2)}\sum_{k=1}^{N} s_k p(\mathbf{r}_k)+\int_\Sigma G(\Sigma)p(\Sigma)d\Sigma\ , \tag{8}$$

Here we took $\psi_0=1$, which is a solution of $L(u)=0$ when $q(\mathbf{r})=0$.

We think that the knowledge of C_l's together with the initial M_l's defined in (6) (the latter are uniquely determined by the initial shape of interface) could describe the whole moving shape in many important physical cases. *We consider Eqs. (7) and (8) to be the main result of this work.*

Formula (7) follows immediately from the following considerations: Since

$$\int_{B(t+dt)}\psi_l d^D\mathbf{r}-\int_{B(t)}\psi_l d^D\mathbf{r}=\int_{\Gamma(t)}\psi_l v_n d\Gamma\, dt\ ,$$

we have

$$\frac{d}{dt}\left[\int_{B(t)}\psi_l d^D\mathbf{r}\right]=\int_{\Gamma(t)}\psi_l v_n d\Gamma\ .$$

Further, because of (5), this equals $\int_{\Gamma(t)}(-p\psi_l\partial_n u)d\Gamma$; and, finally, in view of (3), this expression is

$$\frac{d}{dt}\left[\int_{B(t)}\psi_l d^D\mathbf{r}\right] = \int_{\Gamma(t)} p(u\partial_n\psi_l - \psi_l\partial_n u)d\Gamma \ . \tag{9}$$

Applying Gauss's theorem to the vector field $p(u\,\text{grad}\psi_l - \psi_l\text{grad}u)$ we find that the rhs of Eq. (9) is given by

$$\int_B \text{div}[pu\,\text{grad}\psi_l - p\psi_l\text{grad}u]d^D\mathbf{r} + \int_\Sigma p(u\partial_n\psi_l - \psi_l\partial_n u)d\Sigma + \sum_{k=1}^{N}\int_{\gamma_k} p(u\partial_n\psi_l - \psi_l\partial_n u)d\Sigma \ . \tag{10}$$

Here the summation is over the pointlike charges s_k, and γ_k denotes the surface of the infinitesimal hypersphere around the $\mathbf{r}_k$.

Considering the rhs of (10) one can see that (i) the volume integral over B vanishes because of (1); (ii) the surface integral over Σ also vanishes if the outer boundary Σ is far removed from the center and if ψ_l decays at infinity stronger than $1/r^{D-2}$ [when $\psi_l=1$, this integral is not zero but equals $\int_\Sigma G(\Sigma)p(\Sigma)d\Sigma$, as it is in the rhs of (8)]; (iii) the contribution of the first integrand to the surface integral over the γ_k is zero, while the integral from the second term equals

$$-2\frac{\pi^{D/2}}{\Gamma(D/2)}s_k p(\mathbf{r}_k)\psi_l(\mathbf{r}_k) \ ,$$

since $p(\mathbf{r})$ and $\psi_l(\mathbf{r})$ are regular near the $\mathbf{r}_k$ and due to Gauss's theorem.

Thus the rhs of (10) equals the rhs of (7) [or (8) when $q(\mathbf{r})=0$ and $l=0$], so we have obtained the infinite set of the conserved quantities C_l, if the sources and sinks are nonmoving (i.e., when s_k and $\mathbf{r}_k$ are time independent). Moreover, if $q(\mathbf{r})=0$ and no pointlike singularities are in the domain B (i.e., if all $s_k=0$), then *all of the dynamics of the interface* $\Gamma(t)$ *have been reduced to the linear time dependence of only one "moment,"* M_0, which is the volume of the phase B if L is Laplacian. *All higher moments* M_l *are constant in time.* Note also that *all moments* M_l *have a purely geometrical nature,* and thus carry information about the moving shape.

It is worth considering the special case when the operator L is Laplacian ($p=1$, $q=0$) [9]. For $D=2$, if the ψ_l are chosen as $\psi_l=z^{-l}$, where $z=x+iy$, these integrals coincide with those previously found in explicit form [3] via the coefficients of the appropriate conformal map. These are analogs of the Richardson moments, whose invariance was found earlier for the interior Hele-Shaw problem [6].

In the 3D Laplacian case, one can choose the ψ_l to be a set of spherical functions:

$$\psi_l^{(m)} = P_{(l-1)}^{(m)}(\theta)e^{im\phi}/r^l \ . \tag{11}$$

Here, r, θ, and ϕ are the polar coordinates and $P_l^{(m)}(\theta)$ are the associated Legendre polynomials. In this case, we have a clear physical interpretation of the moments $M_l^{(m)}$. Namely, they are the coefficients of the multipole expansion of the Newtonian potential at an arbitrary point of the empty interior domain A, if the potential is created by the mass uniformly occupying the domain B. Thus the question of whether one can recover the moving shape using only the numbers M_l introduced in (7) is now reduced to the classical inverse potential problem [11] for the reconstruction of the shape of a body of constant density from its Newtonian potential. Our case corresponds to the exterior problem (where the potential is given in the empty hollow of the body: in the phase A). We believe that the connection between pattern formation studies and the inverse potential problem is especially important and deserves close attention, but we do not discuss this problem here. Rather we merely note that in 3D (unlike the 2D case) there is no description of a body (with the exception of the sphere) in terms of a finite number of nonzero moments M_l. (For a detailed description of these difficulties see [12].)

Note that it is also possible in the general elliptic case (when L is not Laplacian) to preserve the interpretation of the conserved quantities M_l as coefficients of the orthogonal expansion by choosing the Green's function $G(\mathbf{r},\mathbf{r}_1)$ of the operator L to be the integrand in (7), since $G(\mathbf{r},\mathbf{r}_1)$ satisfies the conditions imposed on the ψ_l if $\mathbf{r}\in B$ and $\mathbf{r}_1\in A$. Using the Green's-function expansion and by choosing the orthogonal set of the eigenfunctions ψ_l of $L(\psi_l)=0$ bounded at infinity and divergent at the origin and the complementary set of eigenfunctions $\tilde{\psi}_l$ of the same equation but with opposite asymptotic behavior, we have

$$U(\mathbf{r}) = \sum_l \tilde{\psi}_l(\mathbf{r})\frac{d}{dt}\left[\int_B \psi_l(\mathbf{r}_1)d^D\mathbf{r}_1\right] \ .$$

We remark that our main result (7) holds also for time-dependent s_k and $\mathbf{r}_k$ [since we never used the time independence of s_k and $\mathbf{r}_k$ in obtaining (7)]. Although the C_l are no longer conserved, the problem is still integrable, since C_l are known functions of time (if the time dependence of s_k and $\mathbf{r}_k$ is given). Thus the moments M_l are easily controlled parameters, namely, they are just primitives of the time-dependent rhs of (7). In this way, one might be able to govern the motion of the interface by the proper choice of the s_k and $\mathbf{r}_k$

It should also be mentioned that there could be a few exceptions among the C_l's (only one in the 3D Laplacian case) for which Eq. (7) is not valid, and thus they may not be conserved. The nonconserved C_l's correspond to the ψ_l's decaying at infinity but at a rate not stronger than $r^{(2-D)}$. For the 3D Laplacian case the only nonconserved quantity is C_1, which corresponds to $\psi_1=r^{-1}$. Since we do not know the time dependence of C_1, one could suppose that the description of the interface is now less complete. However, although C_1 is not conserved, we have the conserved quantity C_0 [see Eq. (8)]. In other words, we think that the set $\{M_1,M_2,M_3,\ldots\}$ describes the shape with the same degree of completeness as the set $\{M_0,M_2,M_3,\ldots\}$. This question concerning nonconserved quantities among the C_l's does not arise for the interior problem when the scalar field u is given in the phase enclosed by the moving interface $\Gamma(t)$ (in our case in A instead of B).

In conclusion we pose several questions that arise from these studies and that we believe merit attention.

(1) Does the relation expressed by (7) really mean complete integrability of the multidimensional growth?

(2) If yes, to which nonlinear evolutionary PDE's do these constants C_l correspond?

(3) Is there a Hamiltonian structure for these systems?

(4) Do finite-time singularities (cusps) exist here, as in the 2D case? If yes, how may surface tension regularize them?

(5) How can one recover the moving shape from the given set of moments M_l?

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